ANALYTICAL EVALUATION OF AN INFINITE INTEGRAL OVER FOUR SPHERICAL BESSEL FUNCTIONS

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ABSTRACT

An infinite integral over four spherical Bessel functions of the form $\int_0^\infty r^2\, j_{\lambda_1}(k_1 r)\, j_{\lambda_2}(k_2 r)\, j_{\lambda_3}(k_1 r)\, j_{\lambda_4}(k_2 r)\, dr$ is analytically evaluated. The result involves a finite sum over an associated Legendre function of integer degree and half integer order with a real argument greater than 1. Evaluation of such functions is discussed.

1. Introduction

Infinite integrals over spherical Bessel functions have always been of interest due to their occurence in nuclear physics [1-10], particle physics [11] and astrophysics [12-13], to mention a few. In this paper, an infinite integral over four spherical Bessel functions of the form

$$\int_0^\infty r^2\, j_{\lambda_1}(k_1 r)\, j_{\lambda_2}(k_2 r)\, j_{\lambda_3}(k_3 r)\, j_{\lambda_4}(k_4 r)\, dr \tag{1.1}$$

is analytically evaluated for the special case when $k_3 = k_1$ and $k_4 = k_2$. The parameters λ_1, λ_2, λ_3 and λ_4 are assumed positive integers. Recently, a generalised form of the integral has been attempted numerically [14], and analytically [15]. However, the analytical evaluation involves a complicated hypergeometric function. In our evaluation, we start by assuming the momenta k_1, k_2, k_3 and k_4 to be positive and form the sides of a quadrilateral, which is a consequence of the conservation of linear momentum in any physical application. The general result of evaluating eq. (1.1) is in the form of a finite sum over 3j and 6j symbols with a finite integral over Legendre functions with a rational function. This integral is evaluated for the special case when k_3=k_1 and k_4=k_2 in appendix A. The final result for this special case involves a finite sum over an associated Legendre function of integer degree, half-integer order and a real argument greater than 1. Appendix B discusses the evaluation of such functions.

2. Evaluating The Infinite Integral Over Four Spherical Bessel Functions

The integral

$$\int_0^\infty r^2\, j_{\lambda_1}(k_1 r)\, j_{\lambda_2}(k_2 r)\, j_{\lambda_3}(k_3 r)\, j_{\lambda_4}(k_4 r)\, dr \tag{2.1}$$

where k_1, k_2, k_3 and k_4 are positive real numbers, can be written as

$$\begin{aligned} &\frac{2}{\pi}\int_0^\infty K^2\, dK \left(\int_0^\infty r^2\, j_{\lambda_1}(k_1 r)\, j_{\lambda_2}(k_2 r)\, j_L(K r)\, dr\right) \\ &\times \left(\int_0^\infty r'^2\, j_{\lambda_3}(k_3 r')\, j_{\lambda_4}(k_4 r')\, j_L(K r')\, dr\right), \end{aligned} \tag{2.2}$$

using the spherical Bessel functions Closure Relation

$$\int_0^\infty K^2\, j_L(Kr)\, j_L(Kr')\, dK \;=\; \frac{\pi}{2r^2}\,\delta(r' - r). \tag{2.3}$$

The value of L is chosen such that it is the smallest value that satisfies

$$|\lambda_1 - \lambda_2| \le L \le \lambda_1 + \lambda_2, \tag{2.4}$$

and

$$|\lambda_3 - \lambda_4| \le L \le \lambda_3 + \lambda_4. \tag{2.5}$$

Previously [7-10] it was shown that

$$\begin{aligned} &\begin{pmatrix} \lambda_1 & \lambda_2 & L \\ 0 & 0 & 0 \end{pmatrix} \int_0^\infty r^2\, j_{\lambda_1}(k_1 r)\, j_{\lambda_2}(k_2 r)\, j_L(K r)\, dr \;=\; \frac{\pi\beta(\Delta)}{4k_1 k_2 K}\, i^{\lambda_1+\lambda_2-L} \\ &\times (2L+1)^{1/2} \left(\frac{k_1}{K}\right)^L \sum_{\mathcal{L}=0}^{L} \binom{2L}{2\mathcal{L}}^{1/2} \left(\frac{k_2}{k_1}\right)^{\mathcal{L}} \sum_l (2l+1) \\ &\times \begin{pmatrix} \lambda_1 & L-\mathcal{L} & l \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_2 & \mathcal{L} & l \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} \lambda_1 & \lambda_2 & L \\ \mathcal{L} & L-\mathcal{L} & l \end{Bmatrix} P_l(\Delta), \end{aligned} \tag{2.6}$$

where $\Delta \;=\; (k_1^2 + k_2^2 - K^2)/2k_1 k_2$ and $\beta(\Delta) = \theta(1-\Delta)\theta(1+\Delta)$ with θ the Heaviside

function in half-maximum convention. $P_l(x)$ is a Legendre polynomial of degree l, $\begin{pmatrix} \lambda_1 & \lambda_2 & L \\ 0 & 0 & 0 \end{pmatrix}$ is a 3j symbol and $\begin{Bmatrix} \lambda_1 & \lambda_2 & L \\ \mathcal{L} & L-\mathcal{L} & l \end{Bmatrix}$ is a 6j symbol which can be found in any standard angular momentum text [17-18]. Similarly

$$
\begin{aligned}
&\begin{pmatrix} \lambda_3 & \lambda_4 & L \\ 0 & 0 & 0 \end{pmatrix} \int_0^\infty r^2\, j_{\lambda_3}(k_3 r)\, j_{\lambda_4}(k_4 r)\, j_L(Kr)\, dr = \frac{\pi\beta(\Delta')}{4k_3k_4K}\, i^{\lambda_3+\lambda_4-L} \\
&\times (2L+1)^{1/2} \left(\frac{k_3}{K}\right)^L \sum_{\mathcal{L}'=0}^{L} \binom{2L}{2\mathcal{L}'}^{1/2} \left(\frac{k_4}{k_3}\right)^{\mathcal{L}'} \sum_{l'} (2l'+1) \\
&\times \begin{pmatrix} \lambda_3 & L-\mathcal{L}' & l' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_4 & \mathcal{L}' & l' \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} \lambda_3 & \lambda_4 & L \\ \mathcal{L}' & L-\mathcal{L}' & l' \end{Bmatrix} P_{l'}(\Delta'),
\end{aligned} \tag{2.7}
$$

where $\Delta' = (k_3^2 + k_4^2 - K^2)/2k_3k_4$. The result for eq. (2.1) is then

$$
\begin{aligned}
&\begin{pmatrix} \lambda_1 & \lambda_2 & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_3 & \lambda_4 & L \\ 0 & 0 & 0 \end{pmatrix} \int_0^\infty r^2\, j_{\lambda_1}(k_1 r)\, j_{\lambda_2}(k_2 r)\, j_{\lambda_3}(k_3 r)\, j_{\lambda_4}(k_4 r)\, dr \\
&= \frac{\pi\,(k_1k_3)^{L-1}\, i^{\lambda_1+\lambda_2+\lambda_3+\lambda_4-2L}}{8k_2k_4} (2L+1) \sum_{\mathcal{L}=0}^{L} \sum_{\mathcal{L}'=0}^{L} \\
&\times \binom{2L}{2\mathcal{L}}^{1/2} \binom{2L}{2\mathcal{L}'}^{1/2} (k_2/k_1)^{\mathcal{L}} (k_4/k_3)^{\mathcal{L}'} \sum_{l} \sum_{l'} (2l+1)\,(2l'+1) \\
&\times \begin{pmatrix} \lambda_1 & L-\mathcal{L} & l \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_3 & L-\mathcal{L}' & l' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_2 & \mathcal{L} & l \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_4 & \mathcal{L}' & l' \\ 0 & 0 & 0 \end{pmatrix} \\
&\times \begin{Bmatrix} \lambda_1 & \lambda_2 & L \\ \mathcal{L} & L-\mathcal{L} & l \end{Bmatrix} \begin{Bmatrix} \lambda_3 & \lambda_4 & L \\ \mathcal{L}' & L-\mathcal{L}' & l' \end{Bmatrix} J(k_1,k_2,k_3,k_4;l,l',L).
\end{aligned} \tag{2.8}
$$

where

$$
J(k_1,k_2,k_3,k_4;l,l',L) \equiv \int_0^\infty \frac{\beta(\Delta)\,\beta(\Delta')}{K^{2L}}\, P_l(\Delta)\, P_{l'}(\Delta')\, dK. \tag{2.9}
$$

This integral can be analytically evaluated for the special case when $k_3 = k_1$ and $k_4 = k_2$, i.e. $\Delta' = \Delta$ as follows:

Transforming the variable of integration in (2.9) from K to Δ the integral becomes

$$J(k_1, k_2, k_1, k_2; l, l', L) = \frac{k_1 k_2}{(2k_1 k_2)^{L+1/2}} \int_{-1}^{1} \frac{P_l(\Delta)\, P_{l'}(\Delta)}{(y - \Delta)^{L+1/2}}\, d\Delta, \tag{2.10}$$

where $y \equiv (k_1^2 + k_2^2)/2k_1 k_2$.

Using appendix A, (2.10) reduces to

$$\begin{aligned} J(k_1, k_2, k_1, k_2; l, l', L) &= \frac{\sqrt{\pi}}{\Gamma(L+1/2)} \frac{\sqrt{k_1 k_2}}{|k_1^2 - k_2^2|^L} \\ &\times \sum_\mu (2\mu + 1)\, \Gamma(L+\mu+1/2) \begin{pmatrix} l & l' & \mu \\ 0 & 0 & 0 \end{pmatrix}^2 P_L^{-\mu-1/2}\left(\frac{k_1^2 + k_2^2}{|k_1^2 - k_2^2|}\right). \end{aligned} \tag{2.11}$$

The associated Legendre function, $P_L^{-\mu-1/2}\left(\frac{k_1^2+k_2^2}{|k_1^2-k_2^2|}\right)$, has an integer degree and a half-integer order with a real argument that is greater than 1. Appendix B discusses the associated Legendre function for such cases and shows closed form expressions for special cases of L. Furthermore, if $\lambda_2 = \lambda_1 \equiv l_1$, $\lambda_4 = \lambda_3 \equiv l_2$, the ideal choice for L is $L = 0$. Eq. (2.1) then becomes

$$\int_0^\infty r^2\, j_{l_1}(k_1 r)\, j_{l_1}(k_2 r)\, j_{l_2}(k_1 r)\, j_{l_2}(k_2 r)\, dr = \frac{\pi}{4} \sum_\mu \begin{pmatrix} l_1 & l_2 & \mu \\ 0 & 0 & 0 \end{pmatrix}^2 \frac{k_<^{\mu-1}}{k_>^{\mu+2}}, \tag{2.12}$$

where $k_<$ ($k_>$) is the smaller (larger) of k_1 and k_2.

3. Conclusions

The integral over four spherical Bessel functions

$$\int_0^\infty r^2\, j_{\lambda_1}(k_1 r)\, j_{\lambda_2}(k_2 r)\, j_{\lambda_3}(k_1 r)\, j_{\lambda_4}(k_2 r)\, dr \tag{3.1}$$

was analytically evaluated for positive integer λ_1, λ_2, λ_3 and λ_4. The resulting expression involved a finite sum over an associated Legendre function with integer degree and half-integer order. An interesting compact result was found upon setting $\lambda_2 = \lambda_1$ and $\lambda_4 = \lambda_3$.

APPENDIX A: Evaluation of the Integral $\int_{-1}^{1} \frac{P_l(\Delta)\, P_{l'}(\Delta)}{(y-\Delta)^{L+1/2}}\, d\Delta$

Starting with the integral from [19], eq. 7.228, page 830

$$\int_{-1}^{1} \frac{P_\mu(x)}{(y-x)^{L+1/2}}\, dx \;=\; \frac{2}{\Gamma(L+1/2)}\, (y^2 \,-\, 1)^{-(L-1/2)/2}\, e^{-i\pi(L-1/2)}\, Q_\mu^{L-1/2}(y), \qquad (A.1)$$

where $Q_\mu^{L-1/2}(y)$ is a Legendre function of the second kind of degree μ and order $L-1/2$. Also, using [19], eq. 8.739, page 1023 gives

$$e^{-i\pi(L-1/2)}\, Q_\mu^{L-1/2}(y) \;=\; \sqrt{\frac{\pi}{2}}\, \frac{\Gamma(L \,+\, \mu \,+\, 1/2)}{(y^2 \,-\, 1)^{1/4}}\, P_L^{-\mu-1/2}(\frac{y}{\sqrt{y^2 \,-\, 1}}). \qquad (A.2)$$

Hence, the itegral in $(A.1)$ becomes

$$\int_{-1}^{1} \frac{P_\mu(x)}{(y-x)^{L+1/2}}\, dx \;=\; \sqrt{2\pi}\, \frac{\Gamma(L+\mu+1/2)}{\Gamma(L+1/2)}\, (y^2 \,-\, 1)^{-L/2}\, P_L^{-\mu-1/2}(\frac{y}{\sqrt{y^2 \,-\, 1}}). \quad (A.3)$$

Multiplying eq. $(A.3)$ by $(2\mu \,+\, 1)\, P_\mu(\Delta)$ and summing over μ results in

$$\begin{aligned} \frac{1}{(y \,-\, \Delta)^{L+1/2}} \;&=\; \sqrt{\frac{\pi}{2}}\, \frac{(y^2 \,-\, 1)^{-L/2}}{\Gamma(L+1/2)} \sum_\mu (2\mu \,+\, 1)\, \Gamma(L+\mu+1/2)\, P_\mu(\Delta) \\ &\times P_L^{-\mu-1/2}(\frac{y}{\sqrt{y^2 \,-\, 1}}), \end{aligned} \qquad (A.4)$$

using

$$\sum_\mu (2\mu \,+\, 1)\, P_\mu(\Delta)\, P_\mu(x) \;=\; 2\,\delta(\Delta \,-\, x). \qquad (A.5)$$

Hence

$$\begin{aligned} \frac{P_l(\Delta)}{(y \,-\, \Delta)^{L+1/2}} \;&=\; \sqrt{\frac{\pi}{2}}\, \frac{(y^2 \,-\, 1)^{-L/2}}{\Gamma(L+1/2)} \sum_\mu (2\mu \,+\, 1)\, \Gamma(L+\mu+1/2) \\ &\times P_L^{-\mu-1/2}(\frac{y}{\sqrt{y^2 \,-\, 1}}) \sum_{L'} (2L' \,+\, 1) \begin{pmatrix} \mu & l & L' \\ 0 & 0 & 0 \end{pmatrix}^2 P_{L'}(\Delta), \end{aligned} \qquad (A.6)$$

using

$$P_\mu^m(\Delta)\, P_\nu^M(\Delta) \;=\; (-1)^{m+M}\sqrt{\frac{(\mu+m)!\,(\nu+M)!}{(\mu-m)!\,(\nu-M)!}}$$

$$\times \sum_{L'} (2L' \,+\, 1)\sqrt{\frac{(L'-m-M)!}{(L'+m+M)!}} \begin{pmatrix} \mu & \nu & L' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mu & \nu & L' \\ m & M & -m-M \end{pmatrix} P_{L'}^{m+M}(\Delta). \tag{A.7}$$

So, multiplying eq. $(A.6)$ by $P_{l'}(\Delta)$ and integrating over Δ from -1 to 1 results in

$$\int_{-1}^{1} \frac{P_l(\Delta)\, P_{l'}(\Delta)}{(y \,-\, \Delta)^{L+1/2}}\, d\Delta \;=\; \sqrt{2\pi}\,\frac{(y^2 \,-\, 1)^{-L/2}}{\Gamma(L+1/2)} \sum_\mu (2\mu \,+\, 1)$$
$$\times\, \Gamma(L+\mu+1/2) \begin{pmatrix} l & l' & \mu \\ 0 & 0 & 0 \end{pmatrix}^2 P_L^{-\mu-1/2}\Big(\frac{y}{\sqrt{y^2 \,-\, 1}}\Big), \tag{A.8}$$

using

$$\int_{-1}^{1} P_{l'}(\Delta)\, P_{L'}(\Delta)\, d\Delta \;=\; \frac{2}{2l'+1}\,\delta_{L',\,l'}. \tag{A.9}$$

APPENDIX B: Associated Legendre Fumctions, $P_l^m(x)$ for $x > 1$

Previously we have shown that associated Legendre functions, $P_l^m(x)$, for integer l and real m can be written as

$$P_l^m(x) = \frac{(-1)^l}{2^l \Gamma(|l-m|+1)} (\frac{1-x}{1+x})^{m/2} \frac{d^l}{dx^l} [(1-x^2)^l (\frac{1+x}{1-x})^m], \qquad (B.1)$$

for real $-1 < x < 1$, and the factorial is replaced by the gamma function to allow for non-integer m. The special cases for particular l values were

$$P_0^m(x) = \frac{1}{\Gamma(|m|+1)} (\frac{1+x}{1-x})^{m/2}, \qquad (B.2)$$

$$P_1^m(x) = \frac{1}{\Gamma(|1-m|+1)} (x-m) (\frac{1+x}{1-x})^{m/2}, \qquad (B.3)$$

$$P_2^m(x) = \frac{1}{\Gamma(|2-m|+1)} (3\,x^2 - 3\,xm - 1 + m^2) (\frac{1+x}{1-x})^{m/2}, \qquad (B.4)$$

$$P_3^m(x) = \frac{1}{\Gamma(|3-m|+1)} (15\,x^3 - 15\,x^2 m - 9\,x + 6\,xm^2 + 4\,m - m^3) (\frac{1+x}{1-x})^{m/2}, \quad (B.5)$$

$$\begin{aligned} P_4^m(x) = & \frac{1}{\Gamma(|4-m|+1)} (105\,x^4 - 105\,x^3 m - 90\,x^2 + 45\,x^2 m^2 + 55\,xm \\ & - 10xm^3 + 9 - 10\,m^2 + m^4) (\frac{1+x}{1-x})^{m/2}, \end{aligned} \qquad (B.6)$$

Now, by comparing equations 8.702, page 1014 and 8.704, page 1015 of [19], one finds that for real $x > 1$

$$P_l^m(x) = \frac{(-1)^l}{2^l \Gamma(|l-m|+1)} (\frac{x-1}{x+1})^{m/2} \frac{d^l}{dx^l} [(1-x^2)^l (\frac{x+1}{x-1})^m], \qquad (B.7)$$

with the special cases

$$P_0^m(x) = \frac{1}{\Gamma(|m|+1)} (\frac{x+1}{x-1})^{m/2}, \qquad (B.8)$$

$$P_1^m(x) \;=\; \frac{1}{\Gamma(|1-m|+1)}\;(x-m)\;(\frac{x+1}{x-1})^{m/2}, \qquad (B.9)$$

$$P_2^m(x) \;=\; \frac{1}{\Gamma(|2-m|+1)}\;(3\,x^2-3\,xm-1+m^2)\,(\frac{x+1}{x-1})^{m/2}, \qquad (B.10)$$

$$P_3^m(x) \;=\; \frac{1}{\Gamma(|3-m|+1)}\;(15\,x^3-15\,x^2m-9\,x+6\,xm^2+4\,m-m^3)\,(\frac{x+1}{x-1})^{m/2}, \quad (B.11)$$

$$\begin{aligned} P_4^m(x) \;=\; & \frac{1}{\Gamma(|4-m|+1)}\;(105\,x^4-105\,x^3m-90\,x^2+45\,x^2m^2+55\,xm \\ & -10xm^3+9-10\,m^2+m^4)\,(\frac{x+1}{x-1})^{m/2}, \end{aligned} \qquad (B.12)$$

References

1. R. Anni and L. Taffara, Nuovo Cimento **22A**, 12 (1974).
2. E. Elbaz, J. Meyer and R. Nahabetian, Lett. Nuovo Cimento **10**, 418 (1974).
3. A. Baeza, B. Bilwes, R. Bilwes, J. Dias, J.L. Ferrero and J. Raynal, Nucl. Phys. A **437**, 93 (1985).
4. G. D. White, K. T. R. Davies and P. J. Siemens, Ann. Phys. (NY) **187**, 198 (1988).
5. K. T. R. Davies, M. R. Strayer and G. D. White, J. Phys. G **14**, 961 (1988).
6. K. T. R. Davies, J. Phys. G **14**, 973 (1988).
7. R. Mehrem, J.T. Londergan and M. H. Macfarlane, J. Phys. A **24**, 1435 (1991).
8. R. Mehrem, *Ph.D. Thesis*, (Indiana University 1992)(unpublished).
9. R. Mehrem, J.T. Londergan and G. E. Walker, Phys. Rev. C **48**, 1192 (1993).
10. R. Mehrem, Appl. Math. Comp. **217**, 5360 (2011), arXiv: math-ph/0909.0494, 2010.
11. J. Chen and J. Su, Phys. Rev. D **69**, 076003 (2004).
12. A. Hohenegger, Phys. Rev. D **79**, 063502 (2009), arXiv:0806.3098.
13. A. Hohenegger, *Ph.D. Thesis*, (University of Heidelberg, 2010).
14. J. Van Deun and R. Cools, ACM Trans. Math. Softw. **32**, 580 (2006).
15. C. Crucean, arXiv: math-ph/0912.3659, 2009.
16. R. Mehrem and A. Hohenegger, J. Phys. A **43**, 455204 (2010), arXiv: math-ph/1006.2108, 2010.
17. D.M. Brink and G.R. Satchler, *Angular Momentum* (Oxford University Press, London 1962).

18. A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, 1957).

19. I.S. Gradshteyn and I.M. Rhyzik: *Table of Integrals, Series and Products* (Academic Press, New York, 1994).